

AN OPTIMAL APPROXIMATION OF ROSENBLATT SHEET BY MULTIPLE WIENER INTEGRALS*

GUANGJUN SHEN, QIAN YU

Department of Mathematics, Anhui Normal University, Wuhu 241000, China.

ABSTRACT. Let $Z^{\alpha,\beta}$ be the Rosenblatt sheet with the representation

$$Z^{\alpha,\beta}(t, s) = \int_0^t \int_0^s \int_0^t \int_0^s Q^\alpha(t, y_1, y_2) Q^\beta(s, u_1, u_2) B(dy_1, du_1) B(dy_2, du_2)$$

where B is a Brownian sheet, $\frac{1}{2} < \alpha, \beta < 1$, Q^α and Q^β are the given kernel. In this paper, we construct multiple Wiener integrals of the form

$$\int_0^t \int_0^s \int_0^t \int_0^s [k_1(y_1, y_2)^{-\frac{1}{2}\alpha}(u_1, u_2)^{-\frac{1}{2}\beta} + k_2(y_1 \vee y_2)^{\frac{1}{2}\alpha}(y_1 \wedge y_2)^{-\frac{1}{2}\alpha}|y_1 - y_2|^{\alpha-1} \\ \cdot (u_1 \vee u_2)^{\frac{1}{2}\beta}(u_1 \wedge u_2)^{-\frac{1}{2}\beta}|u_1 - u_2|^{\beta-1}] B(dy_1, du_1) B(dy_2, du_2), \quad k_1, k_2 \geq 0,$$

and obtain an optimal approximation of $Z^{\alpha,\beta}(t, s)$.

1. INTRODUCTION

Self-similar processes are stochastic processes that are invariant in distribution under a suitable scaling of time and space. This property is crucial in applications such as network traffic analysis, mathematical finance, astrophysics, hydrology and image processing. For this reason, their analysis has long constituted an important research direction in probability theory. The Hermite process is an interesting class of self-similar processes with long range dependence, it is given as limits of the so called *Non-Central Limit Theorem* studied in Dobrushin and Major [6], Taqqu [17]. Let us briefly recall the general context.

Denote by $H_j(x)$ the Hermite polynomial of order j defined by

$$H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}, \quad j = 1, 2, \dots$$

with $H_0(x) = 1$, and let the Borel function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $E(g(\xi_0)) = 0$, $E(g(\xi_0)^2) < \infty$ and

$$g(x) = \sum_{j=0}^{\infty} c_j H_j(x), \quad c_j = \frac{1}{j!} E[g(\xi_0) H_j(\xi_0)].$$

The Hermite rank of g is defined by

$$k = \min\{j : c_j \neq 0\},$$

Clearly, $k \geq 1$ since $E[g(\xi_0)] = 0$.

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Let g be a function of Hermite rank k and let $(\xi_n)_{n \in \mathbb{N}}$ be a stationary centered Gaussian sequence with $E(\xi_n^2) = 1$ which exhibits long range dependence in the sense that the correlation function satisfies

$$(1.1) \quad r(n) := E(\xi_0 \xi_n) = n^{\frac{2H-2}{k}} L(n),$$

where $k \geq 1$ is an integer, $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity. Then, the *Non Central Limit Theorem* implies that the stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j),$$

converges, as $n \rightarrow \infty$, in the sense of finite dimensional distributions to the process

$$(1.2) \quad Z_H^k(t) = c(H, k) \int_{\mathbb{R}^k} \int_0^t \left(\prod_{j=1}^k (s - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} \right) ds dB(y_1) \dots dB(y_k),$$

where $x_+ = \max\{x, 0\}$ and the above integral is a Wiener-Itô multiple integral with respect to the standard Brownian motion $(B(y))_{y \in \mathbb{R}}$ excluding the diagonals $\{y_i = y_j, i \neq j\}$, $c(H, k)$ is a positive normalization constant depending only on H and k such that $E(Z_H^k(1))^2 = 1$. The process $(Z_H^k(t))_{t \geq 0}$ is called as *the Hermite process* of order k , it is H self-similar and has stationary increments. The class of Hermite processes includes fractional Brownian motion ($k = 1$) which is the only Gaussian process in this class. Their practical aspects are striking: they provide a wide class of processes from which to model long memory, selfsimilarity, and Hölder-regularity, allowing significant deviation from fractional Brownian motion and other Gaussian processes. Since they are non-Gaussian and self-similar with stationary increments, the Hermite processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. For $k \geq 2$, the process (1.2) is not Gaussian. When $k = 2$, the process (1.2) is known as the Rosenblatt process (see Taqqu [16]). More works for the Hermite process and Rosenblatt process can be found in Albin [2], Leonenko and Ahn [8], Abry and Pipiras [1], Maejima and Tudor [9], Tudor [19], Chronopoulou *et al.* [5], Tudor and Viens [20], Torres and Tudor [18], Shieh and Xiao [15], Pipiras and Taqqu [13], Maejima and Tudor [10, 11], Chen, Sun and Yan [4], Garzón, Torres and Tudor [7], Tudor [21], Yan, Li and Wu [22], Shen, Yin and Zhu [14] and the reference therein.

Motivated by all these results, in this paper, we will prove the optimal approximation theorem of Rosenblatt sheet based on the multiple Wiener integrals of form

$$\begin{aligned} & \int_0^t \int_0^s \int_0^t \int_0^s [k_1(y_1, y_2)^{-\frac{1}{2}\alpha} (u_1, u_2)^{-\frac{1}{2}\beta} + k_2(y_1 \vee y_2)^{\frac{1}{2}\alpha} (y_1 \wedge y_2)^{-\frac{1}{2}\alpha} |y_1 - y_2|^{\alpha-1} \\ & \quad \cdot (u_1 \vee u_2)^{\frac{1}{2}\beta} (u_1 \wedge u_2)^{-\frac{1}{2}\beta} |u_1 - u_2|^{\beta-1}] B(dy_1, du_1) B(dy_2, du_2), \quad t, s > 0 \end{aligned}$$

with $k_1, k_2 > 0$. Recall that Rosenblatt sheet with parameter $\frac{1}{2} < \alpha, \beta < 1$ admits an integral representation of the form (see Tudor [21]), for $t \in [0, T], s \in [0, S]$

$$\begin{aligned} Z^{\alpha, \beta}(t, s) &= d_\alpha d_\beta \int_0^t \int_0^s \int_0^t \int_0^s \int_{y_1 \vee y_2}^t \frac{\partial K^{\alpha'}}{\partial x}(x, y_1) \frac{\partial K^{\alpha'}}{\partial x}(x, y_2) dx \\ &\quad \times \int_{u_1 \vee u_2}^s \frac{\partial K^{\beta'}}{\partial y}(y, u_1) \frac{\partial K^{\beta'}}{\partial y}(y, u_2) dy B(dy_1, du_1) B(dy_2, du_2), \\ &:= \int_0^t \int_0^s \int_0^t \int_0^s Q_\alpha(t, y_1, y_2) Q_\beta(s, u_1, u_2) B(dy_1, du_1) B(dy_2, du_2) \end{aligned}$$

where B is a standard Brownian sheet and K is the deterministic kernel given by

$$(1.3) \quad K^H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \quad \text{for } t > s,$$

with $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$, $B(\cdot, \cdot)$ represents the Beta function, $H' = \frac{H+1}{2}$ and $d_H = \frac{1}{H+1} \sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant. Denote

$$Q_H(t, y_1, y_2) = d_H 1_{[0, t]}(y_1) 1_{[0, t]}(y_2) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial x}(x, y_1) \frac{\partial K^{H'}}{\partial x}(x, y_2) dx.$$

In general, for every Borel measurable function $\zeta \in L^2([0, T]^2 \times [0, S]^2)$ the stochastic integral

$$M_{t,s}(\zeta) := \int_0^t \int_0^s \int_0^t \int_0^s \zeta(y_1, y_2, u_1, u_2) B(dy_1, du_1) B(dy_2, du_2), t \in [0, T], s \in [0, S]$$

is well-defined, and the optimal approximation problem is to estimate

$$(1.4) \quad \inf_{\zeta \in L^2([0, T]^2 \times [0, S]^2)} \sup_{t \in [0, T], s \in [0, S]} E(Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta))^2.$$

Noting that if the above minimum is attained at the function ζ^* , then $\zeta^* > 0$ a.e.. In fact, we have

$$\begin{aligned} E[Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta)]^2 &= \frac{1}{2} t^{2\alpha} s^{2\beta} + 2 \int_0^t \int_0^s \int_0^t \int_0^s \zeta^2(y_1, y_2, u_1, u_2) du_2 dy_2 du_1 dy_1 \\ &\quad - 4 \int_0^t \int_0^s \int_0^t \int_0^s Q^\alpha(t, y_1, y_2) Q^\beta(s, u_1, u_2) \zeta(y_1, y_2, u_1, u_2) du_2 dy_2 du_1 dy_1 \end{aligned}$$

for all $t, s \geq 0$.

If $\zeta^*(y_1, y_2, u_1, u_2) \leq 0$, then

$$\sup_{t \in [0, T], s \in [0, S]} E[Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta^*)]^2 \geq \sup_{t \in [0, T], s \in [0, S]} E[Z^{\alpha, \beta}(t, s) - M_{t,s}(|\zeta^*|)]^2.$$

This gives the contradiction. Hence, we can assume that $k_1, k_2 > 0$ and study the optimal approximation problem

$$(1.5) \quad \inf_{\zeta \in \mathcal{K}} \sup_{t \in [0, T], s \in [0, S]} E[Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta)]^2$$

where

$$\mathcal{K} = \{\zeta(y_1, y_2, u_1, u_2) = k_1(y_1, y_2)^{-\frac{1}{2}\alpha}(u_1, u_2)^{-\frac{1}{2}\beta} + k_2(y_1 \vee y_2)^{\frac{1}{2}\alpha}(y_1 \wedge y_2)^{-\frac{1}{2}\alpha}|y_1 - y_2|^{\alpha-1} \\ \cdot (u_1 \vee u_2)^{\frac{1}{2}\beta}(u_1 \wedge u_2)^{-\frac{1}{2}\beta}|u_1 - u_2|^{\beta-1}, k_1, k_2 > 0\},$$

since $Q_H(t, y_1, y_2) \leq C_{H,T}\{(y_1, y_2)^{-\frac{1}{2}H} + (y_1 \vee y_2)^{\frac{1}{2}H}(y_1 \wedge y_2)^{-\frac{1}{2}H}|y_1 - y_2|^{H-1}\}$. For $k \in \mathcal{K}$, denote

$$f(t, s, k_1, k_2) := 2E[Z^{\alpha,\beta}(t, s) - M_{t,s}(\zeta)]^2, t, s \geq 0.$$

The similar approximation for the fractional Brownian motion and Rosenblatt process are first considered by Banna and Mishura [3], Mishura and Banna [12] and Yan, Li and Wu [22], respectively.

The rest of this paper is organized as follows. Section 2 give the representation of the function $f(t, s, k_1, k_2) = 2E[Z^{\alpha,\beta}(t, s) - M_{t,s}(\zeta)]^2$ for $\zeta \in \mathcal{K}$. In Section 3, we consider the function $\sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2)$ in the compact rectangle interval $[0, T] \times [0, S]$. In Section 4 and Section 5, we consider the optimal approximation in the two case $\Delta \leq 0$ and $\Delta > 0$, respectively. Two special cases be considered in Section 6.

2. THE REPRESENTATION OF $f(t, s, k_1, k_2)$

In this section, we will give the representation of $f(t, s, k_1, k_2) = 2E[Z^{\alpha,\beta}(t, s) - M_{t,s}(\zeta)]^2$ for $\zeta \in \mathcal{K}$.

Theorem 2.1. *Let*

$$a(k_2) = 1 + \frac{4k_2^2}{\alpha\beta}B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1) - \frac{8k_2}{\alpha\beta}C_2(\alpha)C_2(\beta),$$

$$b(k_2) = C_1(\alpha)C_1(\beta) - 4k_2B(1-\alpha, \alpha)B(1-\beta, \beta),$$

where $C_1(\alpha) = d_\alpha c_\alpha^2 B^2(1-\alpha, \frac{1}{2}\alpha)$, and

$$C_2(\alpha) = d_\alpha c_\alpha^2 \int_0^1 \int_0^s r^{-\alpha}(1-s)^{\frac{1}{2}\alpha-1}(1-r)^{\frac{1}{2}\alpha-1}(s-r)^{\alpha-1} dr ds,$$

for all $k_1, k_2 \geq 0$ and $\frac{1}{2} < \alpha, \beta < 1$. Then we have

$$f(t, s, k_1, k_2) = a(k_2)t^{2\alpha}s^{2\beta} - 8k_1b(k_2)ts + 4k_1^2 \frac{t^{2-2\alpha}s^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2}, \quad t \in [0, T], s \in [0, S].$$

Remark 1. *As an immediate result we have $a(k_2) \geq 0$ and*

$$(2.1) \quad b^2(k_2)(1-\alpha)^2(1-\beta)^2 \leq \frac{1}{4}a(k_2)$$

for all $\alpha, \beta \in (\frac{1}{2}, 1)$, since $f(t, s, k_1, k_2) \geq 0$. Notice that $a(k_2)$ is also a quadratic equation in k_2 , we get

$$\frac{4}{\alpha\beta}C_2^2(\alpha)C_2^2(\beta) \leq B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)$$

for all $\frac{1}{2} < \alpha, \beta < 1$.

2. Using the constant C_1, C_2 we give the main results and at the end of this paper we give the numerical of these constants (see Figure 1, 2, 3, 4.)

Proof. It is easy to calculate that

$$\begin{aligned}
& \int_0^t \int_0^s \int_0^t \int_0^s Q^\alpha(t, y_1, y_2) Q^\beta(s, u_1, u_2) (y_1 y_2)^{-\frac{1}{2}\alpha} (u_1 u_2)^{-\frac{1}{2}\beta} du_2 dy_2 du_1 dy_1 \\
&= \int_0^t \int_0^t Q^\alpha(t, y_1, y_2) (y_1, y_2)^{-\frac{1}{2}\alpha} dy_1 dy_2 \int_0^s \int_0^s Q^\beta(s, u_1, u_2) (u_1, u_2)^{-\frac{1}{2}\beta} du_1 du_2 \\
&= d_\alpha c_{\alpha'}^2 \int_0^t \int_0^t \int_{y_1 \vee y_2}^t (y_1 y_2)^{-\alpha} u^\alpha (u - y_1)^{\frac{1}{2}\alpha-1} (u - y_2)^{\frac{1}{2}\alpha-1} du dy_1 dy_2 \\
&\quad \times d_\beta c_{\beta'}^2 \int_0^s \int_0^s \int_{u_1 \vee u_2}^s (u_1 u_2)^{-\beta} u^\beta (u - u_1)^{\frac{1}{2}\beta-1} (u - u_2)^{\frac{1}{2}\beta-1} du dy_1 dy_2 \\
&= d_\alpha c_{\alpha'}^2 d_\beta c_{\beta'}^2 B^2(1 - \alpha, \frac{1}{2}\alpha) B^2(1 - \beta, \frac{1}{2}\beta) ts = C_1(\alpha) C_1(\beta) ts,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \int_0^s \int_0^t \int_0^s Q^\alpha(t, y_1, y_2) Q^\beta(s, u_1, u_2) (y_1 \vee y_2)^{\frac{1}{2}\alpha} (y_1 \wedge y_2)^{-\frac{1}{2}\alpha} |y_1 - y_2|^{\alpha-1} \\
&\quad \cdot (u_1 \vee u_2)^{\frac{1}{2}\beta} (u_1 \wedge u_2)^{-\frac{1}{2}\beta} |u_1 - u_2|^{\beta-1} du_2 dy_2 du_1 dy_1 \\
&= d_\alpha c_{\alpha'}^2 \left[\int_0^t \int_0^u \int_0^u (y_1 y_2)^{-\frac{1}{2}\alpha} u^\alpha (u - y_1)^{\frac{1}{2}\alpha-1} (u - y_2)^{\frac{1}{2}\alpha-1} \right. \\
&\quad \cdot (y_1 \vee y_2)^{\frac{1}{2}\alpha} (y_1 \wedge y_2)^{-\frac{1}{2}\alpha} |y_1 - y_2|^{\alpha-1} dy_1 dy_2 du \left. \right] \\
&\quad \times d_\beta c_{\beta'}^2 \left[\int_0^s \int_0^u \int_0^u (u_1 u_2)^{-\frac{1}{2}\beta} u^\beta (u - u_1)^{\frac{1}{2}\beta-1} (u - u_2)^{\frac{1}{2}\beta-1} \right. \\
&\quad \cdot (u_1 \vee u_2)^{\frac{1}{2}\beta} (u_1 \wedge u_2)^{-\frac{1}{2}\beta} |u_1 - u_2|^{\beta-1} du_1 du_2 du \left. \right] \\
&= \frac{C_2(\alpha)}{\alpha} t^{2\alpha} \frac{C_2(\beta)}{\beta} t^{2\beta}.
\end{aligned}$$

Then for all $t \in [0, T]$, $s \in [0, S]$, we have,

$$\begin{aligned}
& \int_0^t \int_0^s \int_0^t \int_0^s Q^\alpha(t, y_1, y_2) Q^\beta(s, u_1, u_2) \zeta(y_1, y_2, u_1, u_2) du_2 dy_2 du_1 dy_1 \\
&= k_1 C_1(\alpha) C_1(\beta) ts + k_2 \frac{C_2(\alpha) C_2(\beta)}{\alpha \beta} t^{2\alpha} s^{2\beta}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \int_0^t \int_0^s \int_0^t \int_0^s \zeta^2(y_1, y_2, u_1, u_2) du_2 dy_2 du_1 dy_1 \\
&= \int_0^t \int_0^s \int_0^t \int_0^s k_1^2 (y_1 y_2)^{-\alpha} (u_1 u_2)^{-\beta} du_2 dy_2 du_1 dy_2 \\
&\quad + \int_0^t \int_0^s \int_0^t \int_0^s k_2^2 (y_1 \vee y_2)^\alpha (y_1 \wedge y_2)^{-\alpha} |y_1 - y_2|^{2\alpha-2} \\
&\quad \cdot (u_1 \vee u_2)^\beta (u_1 \wedge u_2)^{-\beta} |u_1 - u_2|^{2\beta-2} du_2 dy_2 du_1 dy_2
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t \int_0^s \int_0^t \int_0^s 2k_1 k_2 (y_1 y_2)^{-\frac{1}{2}\alpha} (y_1 \vee y_2)^{\frac{1}{2}\alpha} (y_1 \wedge y_2)^{-\frac{1}{2}\alpha} |y_1 - y_2|^{\alpha-1} \\
& \quad \cdot (u_1 u_2)^{-\frac{1}{2}\beta} (u_1 \vee u_2)^{\frac{1}{2}\beta} (u_1 \wedge u_2)^{-\frac{1}{2}\beta} |u_1 - u_2|^{\beta-1} du_2 dy_2 du_1 dy_2 \\
& := I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= k_1^2 \int_0^t \int_0^t (y_1 y_2)^{-\alpha} dy_1 dy_2 \int_0^s \int_0^s (u_1 u_2)^{-\beta} du_1 du_2 \\
&= \frac{t^{2-2\alpha} s^{2-2\beta}}{(1-\alpha)^2 (1-\beta)^2} k_1^2. \\
I_2 &= k_2^2 \left(\int_0^t \int_0^t (y_1 \vee y_2)^\alpha (y_1 \wedge y_2)^{-\alpha} |y_1 - y_2|^{2\alpha-2} dy_1 dy_2 \right) \\
& \quad \cdot \left(\int_0^s \int_0^s (u_1 \vee u_2)^\beta (u_1 \wedge u_2)^{-\beta} |u_1 - u_2|^{2\beta-2} du_1 du_2 \right) \\
&= k_2^2 \left[\int_0^t \int_0^{y_2} y_2^\alpha y_1^{-\alpha} (y_2 - y_1)^{2\alpha-2} dy_1 dy_2 + \int_0^t \int_{y_2}^t y_1^\alpha y_2^{-\alpha} (y_1 - y_2)^{2\alpha-2} dy_1 dy_2 \right] \\
& \quad \cdot \left[\int_0^s \int_0^{u_2} u_2^\beta u_1^{-\beta} (u_2 - u_1)^{2\beta-2} du_1 du_2 + \int_0^s \int_{u_2}^s u_1^\beta u_2^{-\beta} (u_1 - u_2)^{2\beta-2} du_1 du_2 \right] \\
&= k_2^2 (2 \int_0^t y_2^{2\alpha-1} dy_2 \int_0^1 u^{-\alpha} (1-u)^{2\alpha-2} du) \cdot (2 \int_0^s u_2^{2\beta-1} du_2 \int_0^1 u^{-\beta} (1-u)^{2\beta-2} du) \\
&= \frac{k_2^2}{\alpha\beta} B(1-\alpha, 2\alpha-1) B(1-\beta, 2\beta-1) t^{2\alpha} s^{2\beta}.
\end{aligned}$$

$$\begin{aligned}
I_3 &= 2k_1 k_2 \left(\int_0^t \int_0^t (y_1 y_2)^{-\frac{1}{2}\alpha} (y_1 \vee y_2)^{\frac{1}{2}\alpha} (y_1 \wedge y_2)^{-\frac{1}{2}\alpha} |y_1 - y_2|^{\alpha-1} dy_1 dy_2 \right) \\
& \quad \cdot \left(\int_0^s \int_0^s (u_1 u_2)^{-\frac{1}{2}\beta} (u_1 \vee u_2)^{\frac{1}{2}\beta} (u_1 \wedge u_2)^{-\frac{1}{2}\beta} |u_1 - u_2|^{\beta-1} du_1 du_2 \right) \\
&= 2k_1 k_2 \left(\int_0^t \int_0^{y_2} y_1^{-\alpha} (y_2 - y_1)^{\alpha-1} dy_1 dy_2 + \int_0^t \int_{y_2}^t y_2^{-\alpha} (y_1 - y_2)^{\alpha-1} dy_1 dy_2 \right) \\
& \quad \cdot \left(\int_0^s \int_0^{u_2} u_1^{-\beta} (u_2 - u_1)^{\beta-1} du_1 du_2 + \int_0^s \int_{u_2}^s u_2^{-\beta} (u_1 - u_2)^{\beta-1} du_1 du_2 \right) \\
&= 2k_1 k_2 (2 \int_0^t dy_1 \int_0^1 u^{-\alpha} |1-u|^{\alpha-1} du) \cdot (2 \int_0^s du_1 \int_0^1 u^{-\beta} |1-u|^{\beta-1} du) \\
&= 8k_1 k_2 B(1-\alpha, \alpha) B(1-\beta, \beta) ts.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_0^t \int_0^s \int_0^t \int_0^s \zeta^2(y_1, y_2, u_1, u_2) du_2 dy_2 du_1 dy_1 = \frac{k_1^2}{(1-\alpha)^2 (1-\beta)^2} t^{2-2\alpha} s^{2-2\beta} \\
& + \frac{k_2^2}{\alpha\beta} B(1-\alpha, 2\alpha-1) B(1-\beta, 2\beta-1) t^{2\alpha} s^{2\beta} + 8k_1 k_2 B(1-\alpha, \alpha) B(1-\beta, \beta) ts,
\end{aligned}$$

for all $\zeta \in \mathcal{K}$. It follows that

$$\begin{aligned}
f(t, s, k_1, k_2) &= 2E(Z^{\alpha, \beta}(t, s) - M_{t, s}(\zeta))^2 \\
&= t^{2\alpha} s^{2\beta} + 4 \int_0^t \int_0^s \int_0^t \int_0^s \zeta^2(y_1, y_2, u_1, u_2) du_2 dy_2 du_1 dy_1 \\
&\quad - 8 \int_0^t \int_0^s \int_0^t \int_0^s Q^\alpha(t, y_1, y_2) Q^\beta(s, u_1, u_2) \zeta(y_1, y_2, u_1, u_2) du_2 dy_2 du_1 dy_1 \\
&= a(k_2) t^{2\alpha} s^{2\beta} - 8k_1 b(k_2) ts + 4k_1^2 \frac{t^{2-2\alpha} s^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2}.
\end{aligned}$$

This completes the proof. \square

3. THE MAXIMUM VALUE OF $f(t, s, k_1, k_2)$

In this section, in order to obtain the optimal approximation with $k_1, k_2 > 0$. We need to find the maximum value point $P_0(t_0, s_0)$ in the open rectangle interval $(0, T) \times (0, S)$, and the maximum value point at the boundary of rectangle interval $[0, T] \times [0, S]$. Thus, we can get the $\sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2)$.

Lemma 3.1. *The function $f(t, s, k_1, k_2)$ at the open rectangle interval $(0, T) \times (0, S)$ can't get the maximum value point $P_0(t_0, s_0)$, $t_0 \in (0, T)$, $s_0 \in (0, S)$.*

Proof. If the function $f(t, s, k_1, k_2)$ have maximum value point $P_0(t_0, s_0)$, $t_0 \in (0, T)$, $s_0 \in (0, S)$, then $P_0(t_0, s_0)$ must be the stagnation point of the function

$$(t, s) \mapsto f(t, s, k_1, k_2).$$

That is,

$$\frac{\partial f(t, s, k_1, k_2)}{\partial t} \Big|_{P_0(t_0, s_0)} = 0, \quad \frac{\partial f(t, s, k_1, k_2)}{\partial s} \Big|_{P_0(t_0, s_0)} = 0,$$

Solving equations set:

$$\begin{aligned}
\frac{\partial f(t, s, k_1, k_2)}{\partial t} &= 2t^{1-2\alpha} s^{2-2\beta} (\alpha a(k_2) t^{4\alpha-2} s^{4\beta-2} \\
&\quad - 4k_1 b(k_2) t^{2\alpha-1} s^{2\beta-1} + \frac{4k_1^2}{(1-\alpha)(1-\beta)^2}) = 0, \\
\frac{\partial f(t, s, k_1, k_2)}{\partial s} &= 2t^{2-2\alpha} s^{1-2\beta} (\beta a(k_2) t^{4\alpha-2} s^{4\beta-2} \\
&\quad - 4k_1 b(k_2) t^{2\alpha-1} s^{2\beta-1} + \frac{4k_1^2}{(1-\alpha)^2(1-\beta)}) = 0.
\end{aligned}$$

Let $x = t^{2\alpha-1} s^{2\beta-1}$, elementary calculation can obtain

$$x = \frac{k_1}{b(k_2)(1-\alpha)^2(1-\beta)^2},$$

and

$$\sqrt{a(k_2)} = 2b(k_2)(1-\alpha)(1-\beta).$$

Hence, we have

$$f(t_0, s_0, k_1, k_2) = t_0 s_0 [a(k_2) \frac{k_1}{b(k_2)(1-\alpha)^2(1-\beta)^2} - 4k_1 b(k_2)] = 0.$$

But $f(\varepsilon, \varepsilon, 0, 0) = \varepsilon^2 > 0$, for any $\varepsilon > 0$. That is contradiction. Therefore, the function $f(t, s, k_1, k_2)$ at the open rectangle interval $(0, T) \times (0, S)$ can't get the maximum value point $P_0(t_0, s_0)$, $t_0 \in (0, T)$, $s_0 \in (0, S)$. \square

At the boundary $0 \times [0, S]$, and $[0, T] \times 0$, we have $f(t, s, k_1, k_2) = 0$. So, we need to consider the case of extremum value point on the boundary $T \times [0, S]$ and $[0, T] \times S$, with $k_1, k_2 > 0$. We only think about the case of the boundary $T \times [0, S]$. In the same way, we can get case of the boundary $[0, T] \times S$.

It follows from Theorem 2.1, we know that

$$(3.1) \quad f(T, s, k_1, k_2) = a(k_2)T^{2\alpha}s^{2\beta} - 8k_1b(k_2)Ts + 4k_1^2 \frac{T^{2-2\alpha}s^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2},$$

for all $s \in [0, S]$. Differentiating (3.1) with respect to s leads to

$$(3.2) \quad f_s(T, s, k_1, k_2) = 2\beta a(k_2)T^{2\alpha}s^{2\beta-1} - 8k_1b(k_2)T + \frac{8k_1^2}{(1-\alpha)^2(1-\beta)^2}T^{2-2\alpha}s^{1-2\beta}.$$

Let $f_s(T, s, k_1, k_2) = 0$ and $x = k_1s^{1-2\beta}$, which implies that

$$(3.3) \quad F(x) := \beta a(k_2)T^{2\alpha} - 4b(k_2)Tx + \frac{4T^{2-2\alpha}x^2}{(1-\alpha)^2(1-\beta)^2} = 0$$

and the discriminant Δ of the quadratic function $F(x)$ is

$$\Delta = 16T^2((b(k_2))^2 - \frac{\beta a(k_2)}{(1-\alpha)^2(1-\beta)^2}).$$

If $\Delta \leq 0$, then

$$(3.4) \quad \begin{aligned} \sup_{s \in [0, S]} f(T, s, k_1, k_2) &= f(T, S, k_1, k_2) \\ &= T^{2\alpha}S^{2\beta}a(k_2) - 8k_1b(k_2)TS + 4k_1^2 \frac{T^{2-2\alpha}S^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2}. \end{aligned}$$

Using the same method, on the boundary $[0, T] \times S$, we have

$$\sup_{t \in [0, T]} f(t, S, k_1, k_2) = f(T, S, k_1, k_2).$$

If $\Delta > 0$, then the equation (3.3) has two real roots as follows

$$x_1 = \frac{(1-\alpha)^2(1-\beta)}{2T^{1-2\alpha}}(b(k_2) + \sqrt{b^2(k_2) - \frac{\beta a(k_2)}{(1-\alpha)^2(1-\beta)^2}}),$$

and

$$x_2 = \frac{(1-\alpha)^2(1-\beta)}{2T^{1-2\alpha}}(b(k_2) - \sqrt{b^2(k_2) - \frac{\beta a(k_2)}{(1-\alpha)^2(1-\beta)^2}}),$$

which says $s_1 = k_1^{\frac{1}{2\alpha-1}}x_1^{-\frac{1}{2\alpha-1}}$, and $s_2 = k_1^{\frac{1}{2\alpha-1}}x_2^{-\frac{1}{2\alpha-1}}$ are two stagnation points of the function $s \mapsto f(T, s, k_1, k_2)$. Hence, s_1, s_2 are the points of local maximum and

minimum, respectively, since the monotonicity of the function $s \mapsto f(T, s, k_1, k_2)$. This implies that

$$(3.5) \quad \begin{aligned} \sup_{s \in [0, S]} f(T, s, k_1, k_2) &= f(T, S, k_1, k_2), & \text{if } s_1 \geq S, \\ \sup_{s \in [0, S]} f(T, s, k_1, k_2) &= \max\{f(T, s_1, k_1, k_2), f(T, S, k_1, k_2)\}, & \text{if } s_1 < S. \end{aligned}$$

Using the same method, on the boundary $[0, T] \times S$, we can find a t_1 , such that

$$(3.6) \quad \begin{aligned} \sup_{t \in [0, T]} f(t, S, k_1, k_2) &= f(T, S, k_1, k_2), & \text{if } t_1 \geq T, \\ \sup_{t \in [0, T]} f(t, S, k_1, k_2) &= \max\{f(t_1, S, k_1, k_2), f(T, S, k_1, k_2)\}, & \text{if } t_1 < T. \end{aligned}$$

4. THE OPTIMAL APPROXIMATION, CASE $\Delta \leq 0$

Theorem 4.1. *If $\Delta \leq 0$, then we have*

$$\inf_{\zeta \in \mathcal{K}} \sup_{T \times [0, S]} f(t, s, k_1, k_2) = T^{2\alpha} S^{2\beta} a(k_2^*) - 8k_1^* b(k_2^*) TS + 4(k_1^*)^2 \frac{T^{2-2\alpha} S^{2-2\beta}}{(1-\alpha)^2 (1-\beta)^2},$$

where

$$\mathcal{K} = \{ \zeta(y_1, y_2, u_1, u_2) = k_1^*(y_1, y_2)^{-\frac{1}{2}\alpha} (u_1, u_2)^{-\frac{1}{2}\beta} + k_2^*(y_1 \vee y_2)^{\frac{1}{2}\alpha} (y_1 \wedge y_2)^{-\frac{1}{2}\alpha} |y_1 - y_2|^{\alpha-1} \\ \cdot (u_1 \vee u_2)^{\frac{1}{2}\beta} (u_1 \wedge u_2)^{-\frac{1}{2}\beta} |u_1 - u_2|^{\beta-1}, k_1, k_2 > 0 \}$$

and (k_1^*, k_2^*) is the stagnation point of the function

$$(k_1, k_2) \mapsto f(T, S, k_1, k_2),$$

here

$$(4.1) \quad k_1^* = \frac{4B(1-\alpha, \alpha)B(1-\beta, \beta)C_2(\alpha)C_2(\beta) - B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)C_1(\alpha)C_1(\beta)}{8\alpha\beta(1-\alpha)^2(1-\beta)^2B^2(1-\alpha, \alpha)B^2(1-\beta, \beta) - B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)} \\ \times (1-\alpha)^2(1-\beta)^2T^{2\alpha-1}S^{2\beta-1},$$

$$(4.2) \quad k_2^* = \frac{C_2(\alpha)C_2(\beta) - 4\alpha\beta(1-\alpha)^2(1-\beta)^2B(1-\alpha, \alpha)B(1-\beta, \beta)C_1(\alpha)C_1(\beta)}{B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1) - 8\alpha\beta(1-\alpha)^2(1-\beta)^2B^2(1-\alpha, \alpha)B^2(1-\beta, \beta)}.$$

Proof. It follows from equation (3.4), when $\Delta \leq 0$,

$$\sup_{s \in [0, S]} f(T, s, k_1, k_2) = T^{2\alpha} S^{2\beta} a(k_2) - 8k_1 b(k_2) TS + 4(k_1)^2 \frac{T^{2-2\alpha} S^{2-2\beta}}{(1-\alpha)^2 (1-\beta)^2}$$

for all $k_1, k_2 \geq 0$. Let now (k_1^*, k_2^*) be the stagnation point of the function

$$(k_1, k_2) \mapsto f(T, S, k_1, k_2).$$

An elementary calculation can obtain k_1^*, k_2^* which can be denoted by (4.1) and (4.2), and the Hessian matrix \mathbf{H} as follows

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f(T, S, k_1, k_2)}{\partial k_1^2} & \frac{\partial^2 f(T, S, k_1, k_2)}{\partial k_1 \partial k_2} \\ \frac{\partial^2 f(T, S, k_1, k_2)}{\partial k_2 \partial k_1} & \frac{\partial^2 f(T, S, k_1, k_2)}{\partial k_2^2} \end{pmatrix}$$

and

$$\begin{aligned}\frac{\partial^2 f(T, S, k_1 k_2)}{\partial k_1^2} &= \frac{8T^{2-2\alpha}S^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2} > 0, \\ \frac{\partial^2 f(T, S, k_1 k_2)}{\partial k_1 \partial k_2} &= \frac{\partial^2 f(T, S, k_1 k_2)}{\partial k_2 \partial k_1} \\ &= 32TSB(1-\alpha, \alpha)B(1-\beta, \beta), \\ \frac{\partial^2 f(T, S, k_1 k_2)}{\partial k_2^2} &= \frac{8T^{2\alpha}S^{2\beta}}{\alpha\beta}B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1).\end{aligned}$$

So,

$$|\mathbf{H}| = 64T^2S^2\left(\frac{B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)}{\alpha(1-\alpha)^2\beta(1-\beta)^2} - 16B^2(1-\alpha, \alpha)B^2(1-\beta, \beta)\right) > 0$$

for all $\frac{1}{2} < \alpha, \beta < 1$, since $B^2(1-\alpha, \alpha) < \frac{B(1-\alpha, 2\alpha-1)}{1-\alpha}$. This means that the minimal value of $(k_1, k_2) \mapsto f(T, S, k_1, k_2)$ is achieved at point (k_1^*, k_2^*) . This completes the proof. \square

5. THE OPTIMAL APPROXIMATION, CASE $\Delta > 0$

Lemma 5.1. *If $\Delta > 0$, then we have*

$$s_1(k_1^*, k_2^*) < S,$$

and

$$s_2(k_1^*, k_2^*) < S,$$

where (k_1^*, k_2^*) is the stagnation point of the function $(k_1, k_2) \mapsto \sup_{s \in [0, S]} f(T, s, k_1, k_2)$.

Proof. We split the proof in two steps.

Step one. It is easy to obtain that,

$$(5.1) \quad \frac{2}{s_1^{2\beta-1}} > \frac{1}{s_1^{2\beta-1}} + \frac{1}{s_2^{2\beta-1}} = \frac{x_1 + x_2}{k_1} = \frac{b(k_2)(1-\alpha)^2(1-\beta)}{T^{1-2\alpha}k_1}.$$

since $s_2 > s_1$. Hence, when $k_1 = k_1^*$ and $k_2 = k_2^*$,

$$\begin{aligned}(5.2) \quad \frac{b(k_2^*)(1-\alpha)^2(1-\beta)}{T^{1-2\alpha}k_1^*} &= \frac{(C_1(\alpha)C_1(\beta) - 4k_2^*B(1-\alpha, \alpha)B(1-\beta, \beta))(1-\alpha)^2(1-\beta)}{T^{1-2\alpha}k_1^*} \\ &= \frac{S^{1-2\beta}}{1-\beta} > \frac{2}{S^{2\beta-1}}\end{aligned}$$

for $\beta \in (\frac{1}{2}, 1)$. This proves that $s_1(k_1^*, k_2^*) < S$.

Step two.

$$\begin{aligned}\frac{\partial f(T, s, k_1^*, k_2^*)}{\partial s} &= 2\beta a(k_2^*)T^{2\alpha}s^{2\beta-1} - 8k_1^*b(k_2^*)T + \frac{8(k_1^*)^2}{(1-\alpha)^2(1-\beta)}T^{2-2\alpha}s^{1-2\beta} \\ &= 2\beta a(k_2^*)T^{2\alpha}s^{2\beta-1} - 8\frac{b(k_2^*)(1-\alpha)^2(1-\beta)^2}{T^{1-2\alpha}s^{1-2\beta}}b(k_2^*)T + \frac{8(\frac{b(k_2^*)(1-\alpha)^2(1-\beta)^2}{T^{1-2\alpha}s^{1-2\beta}})^2}{(1-\alpha)^2(1-\beta)}T^{2-2\alpha}s^{1-2\beta} \\ &= 2\beta T^{2\alpha}s^{2\beta-1}(a(k_2^*) - 4b^2(k_2^*)(1-\alpha)^2(1-\beta)^2),\end{aligned}$$

since,

$$k_1^* = \frac{b(k_2^*)(1-\alpha)^2(1-\beta)^2}{T^{1-2\alpha}S^{1-2\beta}}, \text{ and } 4b^2(k_2^*) \leq \frac{a(k_2^*)}{(1-\alpha)^2(1-\beta)^2}.$$

Hence,

$$\frac{\partial f(T, s, k_1^*, k_2^*)}{\partial s} \Big|_{s=S} > 0.$$

Thus, $S < s_1$ or $S > s_2$. By step one, we know $S > s_1$, so $s_2(k_1^*, k_2^*) < S$. \square

Lemma 5.2. *Denote*

$$h(k_1, k_2) = f(T, s_1, k_1, k_2) - f(T, S, k_1, k_2).$$

Then the equation $h(k_1, k_2) = 0$ has two solutions k_1' and \bar{k}_1 , which satisfy $0 < k_1' < \bar{k}_1$, $\frac{\partial h}{\partial k_1} \Big|_{k_1=k_1'} > 0$ and $\frac{\partial h}{\partial k_1} \Big|_{k_1=\bar{k}_1} = 0$.

Proof. By Theorem 2.1, we have

$$\begin{aligned} f(T, s_1, k_1, k_2) &= T^{2\alpha} s_1^{2\beta} a(k_2) - 8k_1 b(k_2) T s_1 + 4k_1^2 \frac{T^{2-2\alpha} s_1^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2} \\ &= T^{2\alpha} \left(\frac{k_1}{x_1}\right)^{\frac{2\beta}{2\beta-1}} a(k_2) - 8k_1 b(k_2) T \left(\frac{k_1}{x_1}\right)^{\frac{1}{2\beta-1}} + 4k_1^2 \frac{T^{2-2\alpha} \left(\frac{k_1}{x_1}\right)^{\frac{2-2\beta}{2\beta-1}}}{(1-\alpha)^2(1-\beta)^2} \\ &:= k_1^{\frac{\beta}{\beta-1}} \varphi(k_2) = k_1^{\frac{2\beta}{2\beta-1}} \varphi(k_2). \end{aligned}$$

Let $h(k_1, k_2) = 0$, which implies that

$$(5.3) \quad k_1^{\frac{2\beta}{2\beta-1}} \varphi(k_2) = T^{2\alpha} S^{2\beta} a(k_2) - 8k_1 b(k_2) T S + 4k_1^2 \frac{T^{2-2\alpha} S^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2} = f(T, S, k_1, k_2).$$

Differentiating (5.3) with respect to k_1

$$(5.4) \quad \frac{2\beta}{2\beta-1} k_1^{\frac{1}{2\beta-1}} \varphi(k_2) = -8b(k_2) T S + 8k_1 \frac{T^{2-2\alpha} S^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2}.$$

multiplying by $\frac{2\beta}{(2\beta-1)k_1}$ on both side of (5.3) leads to

$$(5.5) \quad \frac{2\beta}{2\beta-1} k_1^{\frac{1}{2\beta-1}} \varphi(k_2) = \frac{2\beta a(k_2) T^{2\alpha} S^{2\beta}}{(2\beta-1)k_1} - \frac{16\beta b(k_2) T S}{2\beta-1} + 8\beta k_1 \frac{T^{2-2\alpha} S^{2-2\beta}}{(2\beta-1)(1-\alpha)^2(1-\beta)^2}.$$

It follows that

$$\begin{aligned} &-8b(k_2) T S + 8k_1 \frac{T^{2-2\alpha} S^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2} \\ &= \frac{2\beta a(k_2) T^{2\alpha} S^{2\beta}}{(2\beta-1)k_1} - \frac{16\beta b(k_2) T S}{2\beta-1} + 8\beta k_1 \frac{T^{2-2\alpha} S^{2-2\beta}}{(2\beta-1)(1-\alpha)^2(1-\beta)^2}. \end{aligned}$$

This implies that

$$(5.6) \quad (4\beta-4)T^{2-2\alpha}S^{2-2\beta}k_1^2 + 4b(k_2)TS(1-\alpha)^2(1-\beta)^2k_1 - \beta a(k_2)(1-\alpha)^2(1-\beta)^2T^{2\alpha}S^{2\beta} = 0.$$

This is a quadratic equation in k_1 with the two roots

$$\begin{aligned}\overline{k_1} &= \frac{b(k_2)(1-\alpha)^2(1-\beta)^2 + \sqrt{D}}{2(1-\beta)T^{1-2\alpha}S^{1-2\beta}}, \\ \underline{k_1} &= \frac{b(k_2)(1-\alpha)^2(1-\beta)^2 - \sqrt{D}}{2(1-\beta)T^{1-2\alpha}S^{1-2\beta}},\end{aligned}$$

since

$$D = b^2(k_2)(1-\alpha)^4(1-\beta)^4 - \beta a(k_2)(1-\beta)^3(1-\alpha)^2 > 0.$$

It is easy to check that $\overline{k_1}$ is the solution to the equation set

$$\begin{cases} h(k_1, k_2) = 0, \\ \frac{\partial h}{\partial k_1}(k_1, k_2) = 0. \end{cases}$$

In the follows, we will prove that $\underline{k_1}$ is not the solution of the equation $h(k_1, k_2) = 0$. In fact,

$$\begin{aligned}h(\underline{k_1}, k_2) &= f(T, s_1, \underline{k_1}, k_2) - f(T, S, \underline{k_1}, k_2) \\ &= x_2^{\frac{2\beta}{2\beta-1}} \varphi(k_2) S^{2\beta} - a(k_2) T^{2\alpha} S^{2\beta} + 8b(k_2) T S^{2\beta} x_2 - \frac{4T^{2-2\alpha}}{(1-\alpha)^2(1-\beta)^2} x_2^2 S^{2\beta} \\ &= S^{2\beta} (x_2^{\frac{2\beta}{2\beta-1}} \varphi(k_2) - a(k_2) T^{2\alpha} + 8b(k_2) T x_2 - \frac{4T^{2-2\alpha}}{(1-\alpha)^2(1-\beta)^2} x_2^2) \\ &= \frac{2\beta-1}{1-\beta} S^{2\beta} \left(\left(\frac{x_2}{x_1} \right)^{\frac{2\beta}{2\beta-1}} (4b(k_2) T x_1 - a(k_2) T^{2\alpha}) - (4b(k_2) T x_2 - a(k_2) T^{2\alpha}) \right),\end{aligned}$$

since

$$\varphi(k_2) = a(k_2) T^{2\alpha} x_1^{-\frac{2\beta}{2\beta-1}} - 8b(k_2) x_1^{\frac{1}{2\beta-1}} + \frac{4T^{2-2\alpha}}{(1-\alpha)^2(1-\beta)^2} x_1^{2\beta-2}.$$

By (3.3), we have

$$x_1 + x_2 = b(k_2) T^{2\alpha-1} (1-\alpha)^2 (1-\beta), \quad x_1 x_2 = \frac{1}{4} \beta a(k_2) T^{4\alpha-2} (1-\alpha)^2 (1-\beta),$$

since x_1, x_2 are the root of equation $\beta a(k_2) T^{2\alpha} - 4b(k_2) T x + \frac{4T^{2-2\alpha} x^2}{(1-\alpha)^2(1-\beta)} = 0$. Thus,

$$\frac{4b(k_2) T x_1}{a(k_2) T^{2\alpha}} = \beta \frac{x_1 + x_2}{x_2}, \quad \frac{4b(k_2) T x_2}{a(k_2) T^{2\alpha}} = \beta \frac{x_1 + x_2}{x_1}.$$

Let $x = \frac{x_2}{x_1} \in (0, 1)$. Then

$$\begin{aligned}& \left(\frac{x_2}{x_1} \right)^{\frac{2\beta}{2\beta-1}} (4b(k_2) T x_1 - a(k_2) T^{2\alpha}) - (4b(k_2) T x_2 - a(k_2) T^{2\alpha}) \\ &= a(k_2) T^{2\alpha} \left(\left(\frac{x_2}{x_1} \right)^{\frac{2\beta}{2\beta-1}} \left(\frac{\beta(x_1 + x_2)}{x_2} - 1 \right) - \left(\frac{\beta(x_1 + x_2)}{x_1} - 1 \right) \right) \\ &= a(k_2) T^{2\alpha} \left(x^{\frac{2\beta}{2\beta-1}} (\beta x^{-1} + \beta - 1) - \beta - \beta x + 1 \right) \\ &:= a(k_2) T^{2\alpha} \phi(x).\end{aligned}$$

It is easy to check $\phi(x) > 0$. In fact, $\phi(0) = 1 - \beta$, $\phi(1) = 0$, and $\phi'(0) = -\beta$, $\phi'(1) = 0$

$$\phi''(x) = \frac{2\beta(1-\beta)}{(2\beta-1)^2} x^{\frac{2-2\beta}{2\beta-1}} (x^{\frac{1-2\beta}{2\beta-1}} - 1) > 0$$

for all $x \in (0, 1)$ since $2\beta > 1$. This shows that the function $\phi(x)$ is convex on $(0, 1)$ and $\phi'(x)$ is increasing strictly on $(0, 1)$, which implies $\phi'(x) < 0$. It follows that $\phi'(x)$ is decreasing strictly on $(0, 1)$ and

$$\phi(x) > \phi(1) = 0$$

for all $x \in (0, 1)$. Thus, $h(\underline{k}_1, k_2) > 0$.

On the other hand, $h(0, k_2) = -a(k_2)T^{2\alpha}S^{2\beta} < 0$, it follows that the equation

$$h(k_1, k_2) = f(T, s_1, k_1, k_2) - f(T, S, k_1, k_2) = 0$$

admits a root, denoted by k_1' , on $(0, \underline{k}_1)$. Noting that the $k_1 \mapsto f(T, s_1, k_1, k_2)$ is convex and increasing, we find that the equation

$$h(k_1, k_2) = f(T, s_1, k_1, k_2) - f(T, S, k_1, k_2) = 0$$

admits two roots at most since the function $k_1 \mapsto f(T, S, k_1, k_2)$ is a quadratic function. Thus, k_1' is unique in $(0, \underline{k}_1)$ and $\frac{\partial h}{\partial k_1}|_{k_1=k_1'} > 0$. \square

Now, we consider $\sup_{s \in [0, S]} f(T, s, k_1, k_2)$ at the case $\Delta > 0$.

Theorem 5.1. *If $\Delta > 0$, k_1' is given in Lemma 5.2. We have*

$$\max\{f(T, s_1, k_1, k_2), f(T, S, k_1, k_2)\} = f(T, S, k_1', k_2).$$

Proof. By Lemma 5.1, we have

$$\sup_{s \in [0, S]} f(T, s, k_1, k_2) = \max\{f(T, s_1, k_1, k_2), f(T, S, k_1, k_2)\}.$$

It follows from Lemma 5.2 that

$$\max\{f(T, s_1, k_1, k_2), f(T, S, k_1, k_2)\} = f(T, s_1, k_1, k_2)1_{\{k_1 > k_1'\}} + f(T, S, k_1, k_2)1_{\{k_1 < k_1'\}},$$

which implies that

$$\max\{f(T, s_1, k_1, k_2), f(T, S, k_1, k_2)\} = f(T, S, k_1', k_2),$$

since $k_1 \mapsto f(T, s_1, k_1, k_2)$ is increasing and $f(T, S, k_1, k_2)$ is decreasing for $k_1 < k_1'$. \square

Remark 3. By Theorem 5.1 and (3.5), we have

$$\sup_{s \in [0, S]} f(T, s, k_1, k_2) = f(T, S, k_1', k_2),$$

where $k_1' \in (0, \underline{k}_1)$.

Use the same method, on the boundary $[0, T] \times S$, there also exists a $k_1'' > 0$, such that

$$\sup_{t \in [0, T]} f(t, S, k_1, k_2) = f(T, S, k_1'', k_2).$$

From all above, we have

$$\sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2) = \max\{f(T, S, k_1', k_2), f(T, S, k_1'', k_2)\}$$

Theorem 5.2. 1) If $f(T, S, k_1', k_2) > f(T, S, k_1'', k_2)$. Then minimal value

$$\inf_{\zeta \in \mathcal{K}} \sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2)$$

is achieved at point (T, S, k_1', k_2') and equals to $f(T, S, k_1', k_2')$.

2) If $f(T, S, k_1', k_2) < f(T, S, k_1'', k_2)$. Then minimal value

$$\inf_{\zeta \in \mathcal{K}} \sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2)$$

is achieved at point (T, S, k_1'', k_2'') and equals to $f(T, S, k_1'', k_2'')$.

Proof. 1). If $f(T, S, k_1', k_2) > f(T, S, k_1'', k_2)$, then

$$\begin{aligned} \sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2) &= f(T, S, k_1', k_2) \\ &= T^{2\alpha} S^{2\beta} a(k_2) - 8k_1' b(k_2) TS + 4(k_1')^2 \frac{T^{2-2\alpha} S^{2-2\beta}}{(1-\alpha)^2(1-\beta)^2} \\ &= \left(\frac{4T^{2\alpha} S^{2\beta}}{\alpha\beta} B(1-\alpha, 2\alpha-1) B(1-\beta, 2\beta-1) \right) k_2^2 \\ &\quad + (32k_1' T S B(1-\alpha, \alpha) B(1-\beta, \beta) - \frac{8C_2(\alpha)C_2(\beta)}{\alpha\beta} T^{2\alpha} S^{2\beta}) k_2 \\ &\quad + T^{2\alpha} S^{2\beta} - 8k_1' T S C_1(\alpha) C_1(\beta) + \frac{4(k_1')^2}{(1-\alpha)^2(1-\beta)^2} T^{2-2\alpha} S^{2-2\beta}. \end{aligned}$$

This is a quadratic equation of k_2 , and $\frac{4T^{2\alpha} S^{2\beta}}{\alpha\beta} B(1-\alpha, 2\alpha-1) B(1-\beta, 2\beta-1) > 0$. It is easy to find that, when

$$k_2' = \frac{C_2(\alpha)C_2(\beta) - 4\alpha\beta k_1' T^{1-2\alpha} S^{1-2\beta} B(1-\alpha, \alpha) B(1-\beta, \beta)}{B(1-\alpha, 2\alpha-1) B(1-\beta, 2\beta-1)},$$

we have

$$\inf_{\zeta \in \mathcal{K}} \sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2) = \inf_{\zeta \in \mathcal{K}} f(T, S, k_1', k_2) = f(T, S, k_1', k_2').$$

2). If $f(T, S, k_1', k_2) < f(T, S, k_1'', k_2)$, we have

$$\begin{aligned} \inf_{\zeta \in \mathcal{K}} \sup_{t \in [0, T], s \in [0, S]} f(t, s, k_1, k_2) &= \inf_{\zeta \in \mathcal{K}} f(T, S, k_1'', k_2) \\ &= f(T, S, k_1'', k_2''), \end{aligned}$$

where

$$k_2'' = \frac{C_2(\alpha)C_2(\beta) - 4\alpha\beta k_1'' T^{1-2\alpha} S^{1-2\beta} B(1-\alpha, \alpha) B(1-\beta, \beta)}{B(1-\alpha, 2\alpha-1) B(1-\beta, 2\beta-1)}.$$

This completes the proof. \square

6. TWO SPECIAL CASES

In this section we consider two special classes of the approximation function $\zeta \in \mathcal{K}$. First, if $k_2 = 0$, we only consider the boundary $T \times [0, S]$.

Theorem 6.1. *Let $\mathcal{K}_1 = \{\zeta(y_1, y_2, u_1, u_2) = k(y_1 y_2)^{-\frac{1}{2}\alpha}(u_1 u_2)^{-\frac{1}{2}\beta}, k > 0\}$.*

(1) If $C_1^2(\alpha)C_1^2(\beta) - \frac{\beta}{(1-\beta)(1-\alpha)^2} \leq 0$, then

$$\inf_{\zeta \in \mathcal{K}_2} \sup_{t \in [0, T], s \in [0, S]} 2E(Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta))^2 = T^{2\alpha} S^{2\beta} (1 - 4C_1^2(\alpha)C_1^2(\beta)(1-\alpha)^2(1-\beta)^2)$$

with $k = \frac{C_1(\alpha)C_1(\beta)(1-\alpha)^2(1-\beta)^2}{T^{1-2\alpha}S^{1-2\beta}}$.

(2) If $C_1^2(\alpha)C_1^2(\beta) - \frac{\beta}{(1-\beta)(1-\alpha)^2} > 0$, then

$$\inf_{\zeta \in \mathcal{K}_2} \sup_{t \in [0, T], s \in [0, S]} 2E(Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta))^2 = f(T, S, k'_1, 0)$$

where $\zeta(y_1, y_2, u_1, u_2) = k'_1(y_1 y_2)^{-\frac{1}{2}\alpha}(u_1 u_2)^{-\frac{1}{2}\beta}$ and k'_1 is the smallest root of the equation $f(T, s_1, k, 0) - f(T, S, k, 0) = 0$.

Proof. For $\zeta \in \mathcal{K}_1$ we have

$$\begin{aligned} 2E(Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta))^2 &= f(t, s, k, 0) \\ &= t^{2\alpha} s^{2\beta} - 8k_1 C_1(\alpha) C_1(\beta) t s + \frac{4(k_1)^2}{(1-\alpha)^2(1-\beta)^2} t^{2-2\alpha} s^{2-2\beta} \end{aligned}$$

and $\Delta_1 = 64k^2 T^2 [C_1^2(\alpha)C_1^2(\beta) - \frac{\beta}{(1-\beta)(1-\alpha)^2}]$, which competes the proof. \square

Second, we consider the case $k_1 = 0$.

Theorem 6.2. *Let*

$$\begin{aligned} \mathcal{K}_2 = \{ \zeta(y_1, y_2, u_1, u_2) &= k(y_1 \vee y_2)^{\frac{1}{2}\alpha}(y_1 \wedge y_2)^{-\frac{1}{2}\alpha} |y_1 - y_2|^{\alpha-1} \\ &\cdot (u_1 \vee u_2)^{\frac{1}{2}\beta}(u_1 \wedge u_2)^{-\frac{1}{2}\beta} |u_1 - u_2|^{\beta-1}, k > 0 \} \end{aligned}$$

we have

$$\begin{aligned} \inf_{\zeta \in \mathcal{K}_2} \sup_{t \in [0, T], s \in [0, S]} 2E(Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta))^2 \\ = (1 - \frac{4C_2^2(\alpha)C_2^2(\beta)}{\alpha\beta B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)}) T^{2\alpha} S^{2\beta} \end{aligned}$$

with $k^ = \frac{C_2(\alpha)C_2(\beta)}{B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)}$.*

Proof. By Theorem 2.1, we have

$$\inf_{\zeta \in \mathcal{K}_2} \sup_{t \in [0, T], s \in [0, S]} 2E(Z^{\alpha, \beta}(t, s) - M_{t,s}(\zeta))^2 = \inf_{\zeta \in \mathcal{K}_2} a(k) T^{2\alpha} S^{2\beta}.$$

The function

$$a(k) = 1 + \frac{4k^2}{\alpha\beta} B(1-\alpha, 2\alpha-1) B(1-\beta, 2\beta-1) - \frac{8k}{\alpha\beta} C_2(\alpha) C_2(\beta)$$

is a quadratic equation in k , then

$$\begin{aligned} \inf_{\zeta \in \mathcal{K}_2} a(k) T^{2\alpha} S^{2\beta} &= a(k^*) T^{2\alpha} S^{2\beta} \\ &= \left(1 - \frac{4C_2^2(\alpha)C_2^2(\beta)}{\alpha\beta B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)}\right) T^{2\alpha} S^{2\beta} \end{aligned}$$

with $k^* = \frac{C_2(\alpha)C_2(\beta)}{B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)}$.
and

$$\begin{aligned} \zeta(y_1, y_2, u_1, u_2) &= \frac{C_2(\alpha)C_2(\beta)(y_1 \vee y_2)^{\frac{1}{2}\alpha}(y_1 \wedge y_2)^{-\frac{1}{2}\alpha}|y_1 - y_2|^{\alpha-1}}{B(1-\alpha, 2\alpha-1)B(1-\beta, 2\beta-1)} \\ &\quad \cdot (u_1 \vee u_2)^{\frac{1}{2}\beta}(u_1 \wedge u_2)^{-\frac{1}{2}\beta}|u_1 - u_2|^{\beta-1} \end{aligned}$$

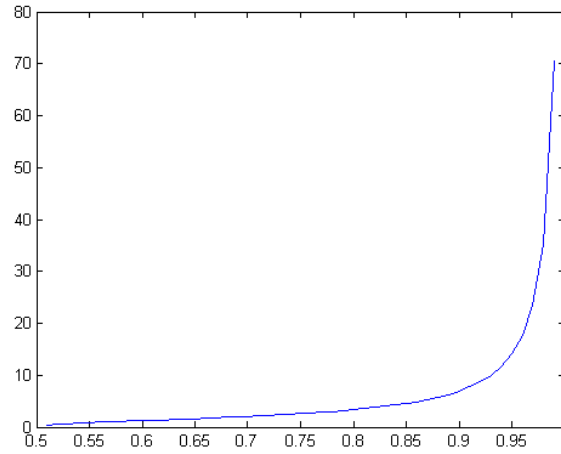
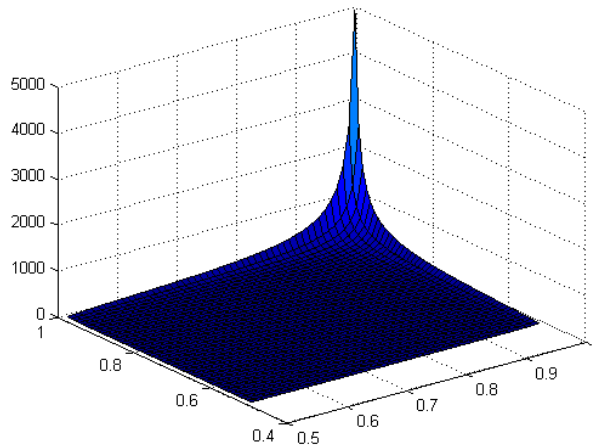
where $y_1, y_2 > 0$ and $u_1, u_2 > 0$. This completes the proof. \square

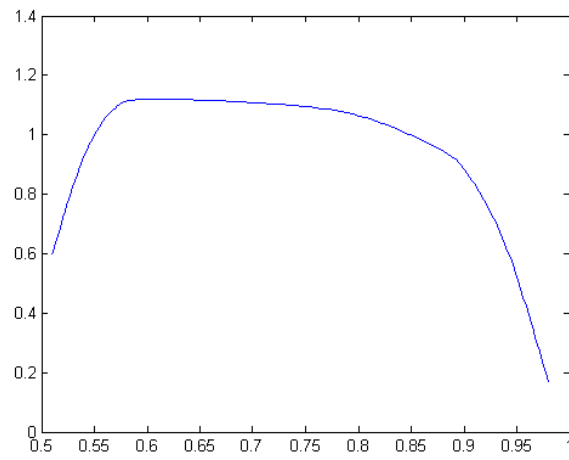
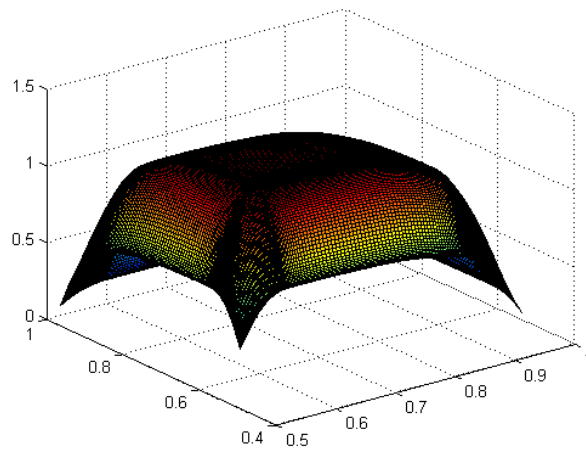
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 FIGURE 1. $\alpha \mapsto C_1(\alpha)$

 FIGURE 2. $(\alpha, \beta) \mapsto C_1(\alpha)C_1(\beta)$

FIGURE 3. $\alpha \mapsto C_2(\alpha)$ FIGURE 4. $(\alpha, \beta) \mapsto C_2(\alpha)C_2(\beta)$